Spring 2015

Supplemental Handouts to support Textbook for the Course

More handouts may be provided.

Check Canvas for Power Points, too

An engineer, physicist, mathematician, and a statistician are all staying at a hotel. In the middle of the night the engineer wakes up to find that his trashcan is on fire. He runs to the sink, fills his ice bucket with water and douses the flames. Then, just to be sure, he runs back to the sink, refills the bucket and dumps more water into the trashcan. With the fire out, he goes back to sleep.

A little while later, the trashcan in the physicist's room spontaneously breaks into flame, waking the physicist. He whips out his slide rule, does some calculations, then runs to the sink, fills his bucket with exactly .75 liters of water, and douses the flames. Having put out the fire, he goes back to sleep.

A few minutes later, the mathematician wakes up to see that his trashcan is on fire. He whips out a piece of paper, scrawls out some equations, then goes back to sleep, comfortable that a solution exists.

Meanwhile, the statistician is running from room to room, lighting trashcans on fire -- he needed more samples.

From Canvas (or this document), go to the link below. There is a 3 hour tutorial (which took Mrs. B about 4 hours – this includes writing this quiz based off of the tutorial) on MATLAB. Click on "Launch Tutorial" to start watching the videos and to do interactive quizzes/lessons.

Note: Do NOT pay for the additional content that is missing (unless you want to). You are not held accountable for the missing content.

There are 2 quizzes that will impact your MATLAB points for the course. No MATLAB assignments can be dropped from your grade.

A. Take Home quiz – worth 30 points
B. In class quiz – worth 15 points

In class quiz (note: the online tutorial will explain how to do this):
On Monday, 1/26, you will do the following on your computers:

Everyone will write their height (in decimals and units of ft) and their shoe size (ladies – give the men's shoe size for you) on the board. Everyone will create an Excel file with the heights and shoe sizes. Import the data into MATLAB. You will then be asked to graph the shoe sizes and label the x and y-axis, along with giving a title to your plot. You will need to create as a scatterplot (not line graph) using MATLAB. You will have 20 minutes to do this in class. Mrs. B will grade the work off of your computer.

Take Home Quiz:
As you watch the tutorials, answer the following questions. Take screen shots of ALL interactive parts of the tutorials. Include the screen shots in a Word document, along with the answers to the following quiz. Email your results to Mrs. B by Sunday, January 25th (midnight). Do NOT wait until Sunday to do this. I recommend doing 30-40 minutes every day. There will be an in class quiz on Monday, January 26th covering this tutorial (see above for more info).

Answer all questions in bullet form (quick and precise answers – some questions can be answered in 1 word)

https://www.mathworks.com/academia/student_center/tutorials/mltutorial_launchpad.html?confirmation_page#

1. What are the 3 data set examples that the Matlab tutorial will cover?
2. What are the names of the 4 windows on the desktop of Matlab?
3. What is the purpose of each window?
4. Take the Matlab quiz, take screen shots of your correct solution.
5. There are two ways to view data – what are they?
6. Which country has a gas price missing – and for what year?

7. Take Matlab quiz about data – take screen shots of your correct answers.
8. Give at least 3 ways to import data into Matlab?
9. Take quiz – give screen shot of your result
10. How do you save your variables so that you can close Matlab and come back later? Where will it be saved?
11. How do you load the variables when you open Matlab?
12. If we call a variable the following name

    cat

    What is wrong with the following command that is calling on the variable name?

    Cat
13. What are the three items that variable names can contain? What must the variable start with?
14. What is used to create a column vector? Row vector?
15. How do you create vectors that are equally spaced so that you don't have to type in every value? For example: Suppose you wanted to deal with ages 18-30. What would be the command to define the years of increments of 1? (Name this command "age")
16. To type in a different increment (suppose you want to go up every 2 years for the age), what would you type? (name this command "age")
17. What does the function "linspace(a,b,n)" mean? Explain, and then state the command for the ages in #15 with 4 points in-between. (name this command "age")
18. To switch from a row to a column vector, what would be typed to make this happen? Continue to use the command name "age" as an example.
19. If you don't want to see the output of the command, what do you type at the end of your command? Use the command name "age" as an example as to what to type.
20. What do you type so that you can learn more about a command? For example, if you wanted to learn about plotting in matlab, what would you type?
21. What would (:,10) extract in a matrice?
22. Take a screen shot of all the plot commands in MATLAB (stop the video when it shows all the commands).
23. How do you label your plots? You can take a screen shot of this as well to include in this quiz (or you can write them out).
24. What is the difference between using a "*" to multiply and ".*"?

    Note: This is SO important with plotting direction fields!

Scripts – we use this concept in diff eq.

25. Whale call model – go through MATLAB and type in the commands to do the Whale call model (don't just watch the video). You will see how to manually type in code.
26. Then – you will learn about the "scripts" option so that you do not have to manually type in code to make one change – again – do this along with the video so that you can see how to do this.
27. To create a script, what do you type at the command prompt?

28. How do you enter notes into your script so that you can have information in

your script (mfile)? I.E. How can you enter “cheat sheet notes” to help you with the mfile?

You do not have to watch the training for “Working with Data Files” (watch if you want)

Under “Multiple Vector Plots”, some of this info will help us with understanding the codes for direction fields.

Under “Logic and Flow Control”, you will watch videos on “Programming”. This is important!

29. What do you type in to obtain a list of key words in MATLAB to program (create code)?

30. What does it mean to publish results? Why would this be useful for a student to do if they don’t have MATLAB at home?

Email the questions, along with the answers and screen shots, to Mrs. B

Intro to Diff Eq

- Suppose that the velocity function, in m/s, can be found by
    - $v(t) = -9.8\ t + 12$
    - $s(0) = 2.3$ (i.e. initial position is 2.3 meters)

Find the position function.

Ordinary Differential Equation (ODE):

Definitions you should know:

Order of ODE:

Linear ODE:

Nonlinear ODE:

Partial Differential Equation:

Ordinary Differential Equation:

Independent vs. Dependent Variable:

Notation:

Implicit Differentiation:

Initial Value Problems:

Uniqueness:

Practice (see PPT)

Matrices (Chapter 9 of Textbook)

NOTES 1: Solve 3x3 Systems

Steps for Solving a 3 by 3 System

1. Write out the 3 equations (with 3 variables)
2. Eliminate 1 variable
3. Use the equation that you DID NOT JUST USE with 1 of the equations JUST USED and eliminate the SAME variable.
4. Use the 2 equations you developed as a 2 by 2 system to solve for the remaining variables
5. Substitute that value back into an equation from the 2 by 2 system and solve for the other variable
6. Substitute both variables into an original equation (3 by 3) to solve for the last variable
7. **Check your answer** by plugging into each equation from the original system and checking that it makes a true statement!

Ex. $\begin{cases} x+y-z=0 \\ x+2y+2z=22 \\ 6x-3y+z=6 \end{cases}$

OYO $\begin{cases} 2x-y+2z=4 \\ x+y-2z=5 \\ 3x-y-2z=15 \end{cases}$

Now – we want to show another way to solve the above, but we first need to introduce them – Matrices!

Matrix – rectangular array of numbers written using rows and columns

- A **matrix** is a rectangular array of numbers arranged in rows and columns, e.g. $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 1 & 7 \end{pmatrix}$.

- A matrix is usually denoted by a bold capital letter, e.g. **A** (or $\underline{A}$ when hand-written).

- The matrix $\begin{pmatrix} 2 & -3 & 0 \\ -1 & 1 & 7 \end{pmatrix}$ has 2 rows and 3 columns and so is described as a 2×3 matrix. The **order** of this particular matrix is said to be 2×3.

- A matrix with a single column, e.g. $\begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix}$, is called a **column vector**. Likewise a matrix with just one row, e.g. (1 -4), is called a **row vector**. We usually label column vectors and row vectors by lower case bold letters, e.g.

$$\mathbf{a} = \begin{pmatrix} 2 & 1 & -9 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -6 \\ 3 \end{pmatrix}.$$

- If a matrix has an equal number of rows and columns, it is called a **square matrix**, e.g. $\begin{pmatrix} 0 & 7 \\ -2 & 4 \end{pmatrix}$ is an example of a 2×2 square matrix.

- A **diagonal matrix** has zeros everywhere except on the **leading diagonal**, e.g.

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \text{ or } \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

- An **identity matrix** is a diagonal matrix that has 1's on the leading diagonal,

  e.g. the 2×2 identity matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the 3×3 identity matrix is

$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Practice how many rows and columns?

$$B = \begin{bmatrix} -1 & -2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & \frac{1}{3} \\ p & 4 \\ 0 & -6 \\ 0.7 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 9 & a & -6 & \frac{2}{3} \end{bmatrix}$$

<u>Identifying elements</u> in a matrix: in Matrix B, b_{12} is the element in row 1, column 2 =________ , b_{23}= _______

c_{41} = ________ c_{12}=__________ c_{21} = _________ d_{12} = _______

$$A = \begin{array}{cc} \text{row}\rightarrow & \begin{bmatrix} x & -7 & 8 & 6 \\ 0 & 6 & y & -3 \\ -4 & 9 & 1 & 2 \end{bmatrix} \\ & \begin{array}{l} \uparrow \\ \text{column} \end{array} \end{array}$$

x is in the__________ row and __________column

y is in the ___________row and __________column

Solving using matrices.

Set up an equation(s) for each example and solve.

Example 1:

$$\begin{bmatrix} x & 7 \\ 15 & -3 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 15 & -3 \end{bmatrix}$$

Example 2: $\begin{bmatrix} 2x-7 & 0 \\ 5 & y+4 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 5 & 12 \end{bmatrix}$

OYO:

$$\begin{bmatrix} 2x & 8 \\ -3 & y \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ -3 & -5 \end{bmatrix}$$

OYO: $\begin{bmatrix} x+4 & -3 \\ 9 & y-3 \end{bmatrix} = \begin{bmatrix} -16 & -3 \\ 9 & 7 \end{bmatrix}$

Two matrices can only be added or subtracted if they have the same size (i.e. they have the same number of rows and columns).

To add (or subtract) two matrices, we add (or subtract) the corresponding entries.

The matrix $k\mathbf{A}$, where k is a scalar (i.e. a single number), is obtained by multiplying all of the entries in matrix $\mathbf{A}$ by k.

Ex) Let $A = \begin{pmatrix} 2 & -6 \\ 3 & 2 \end{pmatrix}$ Find $4A = 4\begin{pmatrix} 2 & -6 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4*2 & 4*-6 \\ 4*3 & 4*2 \end{pmatrix} = \begin{pmatrix} 8 & -24 \\ 12 & 8 \end{pmatrix}$

Ex) Let $B = \begin{pmatrix} 3 & -4 \\ 6 & 7 \end{pmatrix}$ Find $-B = \begin{pmatrix} -3 & 4 \\ -6 & -7 \end{pmatrix}$

Practice Let $A = \begin{pmatrix} 2 & 4 & 2 \\ 3 & 3 & 6 \\ 7 & 0 & -5 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 & 7 \\ 0 & 4 & 6 \\ -1 & -2 & 5 \end{pmatrix} and\_C = \begin{pmatrix} 2 & 4 \\ -6 & 3 \end{pmatrix}$

1) Find 2B

2) Find – 3A

3) Find 2A – B

Remember: we write the dimensions of matrices as r × c

 # columns of [A] = # rows of [B] so that you can match up the entries to complete the matrix multiplication

$A = \begin{bmatrix} 3 & 5 & 7 \\ 4 & 1 & 2 \end{bmatrix} B = \begin{bmatrix} 2 & 3 & 3 & 8 & 4 \\ 9 & 5 & 2 & 0 & 6 \\ 0 & 1 & 6 & 7 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 5 \\ 4 & 1 \\ 5 & 8 \end{bmatrix} D = \begin{bmatrix} 2 & 3 & 3 & 8 & 4 \\ 9 & 5 & 2 & 0 & 6 \\ 0 & 1 & 6 & 7 & 2 \end{bmatrix}$

[A] [B] YES – it can multiply

$2 \times 3 = 3 \times 5$

[C] [D] No – it can't multiply

$3 \times 2 \neq 3 \times 5$

Calculator NOTE: If the 1st matrix does not have the same number of columns as rows of the 2nd matrix there would not be a partner multiply with → the calculator will give you ERR:DIM MISMATCH – meaning your dimensions don't match and you cannot perform this operation (+/-/×/÷)

For two matrices **A** and **B**, the product **AB** is only defined if the number of columns of **A** is the same as the number of rows in **B.**
To calculate **AB** we multiply each row of A by each column of B using the dot product.
We can illustrate this process by an example:

Suppose **A** = $\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 5 \end{pmatrix}$ and **B** = $\begin{pmatrix} 5 & 3 & -2 \\ 4 & 1 & 2 \end{pmatrix}$

then
AB =
$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 5 & 3 & -2 \\ 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1\times5+2\times4 & 1\times3+2\times1 & 1\times(-2)+2\times2 \\ 3\times5+(-1)\times4 & 3\times3+(-1)\times1 & 3\times(-2)+(-1)\times2 \\ 2\times5+5\times4 & 2\times3+5\times1 & 2\times(-2)+5\times2 \end{pmatrix} = \begin{pmatrix} 13 & 5 & 2 \\ 11 & 8 & -8 \\ 30 & 11 & 6 \end{pmatrix}.$$

Example 1: AB

$$\begin{bmatrix} 0 & -1 \\ -1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 5 \end{bmatrix} =$$

WRITE OUT THE DIMENSIONS FIRST TO CHECK THAT IT WILL WORK!!!!!

Example 2:

$$\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & -3 \end{bmatrix} =$$

OYO:

$$\begin{bmatrix} 0 & 1 \\ -1 & 8 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} =$$

OYO: $\begin{bmatrix} 0 & 7 & 3 \\ -2 & 3 & 0 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} =$

Wouldn't it be nice to have a CALCULATOR do this for us? ☺

<u>Finding Inverse of Matrix</u>

Suppose that two (square) matrices **A** and **B** are such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

A is then called the *inverse* of **B** (and **B** is the inverse of **A**). We write $\mathbf{A} = \mathbf{B}^{-1}$ and $\mathbf{B} = \mathbf{A}^{-1}$.

Example: Note that $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

So the inverse of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ and the inverse of $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ is $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$.

How to find the inverse of another matrix

Example: $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

Example: $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Now use calculator to find the inverse

DETERMINANTS

denoted by - det or $|\quad|$

The matrix must be a square matrix in order to find the determinant.

BY HAND

$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$

Example 1: $\begin{vmatrix} 3 & -4 \\ 5 & 3 \end{vmatrix} =$

IN THE CALCULATOR

Example 2: $[B] = \begin{bmatrix} 3 & -2 \\ -4 & 6 \end{bmatrix}$, $\det[B] =$

OYO: $[D] = \begin{bmatrix} 4 & -2 & 0 \\ -3 & 10 & 1 \\ 2 & 6 & -1 \end{bmatrix}$, $|D| =$

OYO: $[c] = \begin{bmatrix} 4 & 2 & -3 \\ 0 & 1 & -4 \end{bmatrix}$, $\det[c] =$

OYO: $[A] = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & -2 \\ 10 & 3 & -1 \end{bmatrix}$, $|A| =$

Another way to find the Inverse of a Matrix

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ when det A $\neq$ 0.

Calculator NOTE: *If you take the inverse when the determinant is zero because it would be dividing by zero → the calculator will give you ERR:SINGULAR MAT – meaning you have a singular matrix*

Example 1:

$$[B] = \begin{bmatrix} 3 & -2 \\ -4 & 6 \end{bmatrix}, \det[B] =$$

$$[B]^{-1} =$$

Ok – let's solve the two problems at the beginning using matrices now.

Ex. $\begin{cases} x + y - z = 0 \\ x + 2y + 2z = 22 \\ 6x - 3y + z = 6 \end{cases}$

OYO $\begin{cases} 2x - y + 2z = 4 \\ x + y - 2z = 5 \\ 3x - y - 2z = 15 \end{cases}$

Let's use your calculator, too.

Calculus of Matrices

Given X(t), find the following:

$$X(t) = \begin{bmatrix} 7t & -3\sin(2t) & -3\cos(4t) \\ -e^{5t}+2 & 5-t^3 & 9t \\ i\sin(t) & e^{it} & 8 \end{bmatrix}$$

dX/dt

$\int X(t)\,dt$

Matrices and Differential Equations

Eigenvalue and Eigenvectors

Definition: Let 'A' be an n x n matrix. A scalar λ (lambda) is called an eigenvalue of A if there is a nonzero vector $\bar{x}$ such that $Ax = \lambda x$.

Such a vector $\bar{x}$ is called an eigenvector of A corresponding to lambda.

Ex) Show that x is an eigenvector and lambda is an eigenvalue

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix} \qquad \lambda = 4$$

Calc III:

- Note: If lambda is an eigenvalue of A, and x is an eigenvector belonging to lambda, any nonzero multiple of x will be an eigenvector.

In Calc III → what are vectors that are scalar multiples of each other?

Are the following eigenvectors? Why or why not?

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 4 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \to \begin{bmatrix} -10 \\ -5 \end{bmatrix} \to \begin{bmatrix} -200 \\ -100 \end{bmatrix}$$

If vectors are scalar multiples of each other, then are they linearly independent or dependent?

Based off of the above, what can you say about the determinant?

So, if we have $Ax = \lambda x$, then we have the following:

Example: Find the eigenvalues and eigenvectors of the following:

I. $\begin{bmatrix} 7 & -4 \\ 5 & -2 \end{bmatrix}$

II. $\begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix}$

III. $\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Now – let's put this all together:

How to solve systems of differential equations using matrices.

A. Write the problem in matrix form.

B. The solution will be of the form $Ce^{\lambda t}$ since we have constant coefficients. (why?)

C. Find the eigenvalues (this will give us λ for part B).

D. Find the eigenvectors.

E. The general solution will be including the eigenvector * e^{λ_1} and e^{λ_2} (why do you think this is true?)

A. Given $\begin{matrix} y_1'(t) = y_1(t) - 3y_2(t) \\ y_2'(t) = y_1(t) + 5y_2(t) \end{matrix}$, find the general solution to the system.

B. Given $\begin{matrix} y_1'(t) = y_1(t) + y_2(t) \\ y_2'(t) = 4y_1(t) + y_2(t) \end{matrix}$, find the general solution to the system.

In Freemat/Matlab, the eigenvalues of the matrix - or the eigenvalues and eigenvectors of a matrix - can be found using the eig command.

Example 1. 2×2 matrix

For example| the solution of the eigenvalue problem with $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ can be determined with the following code.

```
–> A=[3 -2; -1 4]
A =
3 -2
-1  4
-> eig(A)
ans =
2
 5
-> [E,V]=eig(A)
E =
-0.8944   0.7071
 -0.4472  -0.7071
V =
2 0
 0 5
```

Note that the command eig(A) simply returns the eigenvalues of A, but the command [E,V]=eig(A) returns both the eigenvalues and eigenvectors. The eigenvalues are given by the diagonal components of the matrix V and the corresponding eigenvectors are the columns or the matrix E.

Comparing the results with the document from which the examples came[1], we see that the eigenvalue results match, but the eigenvalue results do not. The eigenvalues found in the referenced document are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. H|owever, with eigenvectors it is the ratio of the elements that matters, not the actual values. In Matlab/Freemat, the eigenvectors have been normalised[2], so that their norm is one.

Example 2. 3×3 matrix

For example the solution of the eigenvalue problem with $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$ can be determined with the following code.

```
--> A=[1 -1 0; 1 2 1; -2 1 -1]
A =
1 -1  0
1  2  1
-2 1 -1
--> eig(A)
ans =
-1.0000
1.0000
2.0000
--> [E,V]=eig(A)
E =
-0.1361   -0.7071   -0.5774          -
0.2722  -0.0000    0.5774
0.9526    0.7071    0.5774
V =
-1.0000        0        0
 0    1.0000        0
 0        0    2.0000
```

Practice for matrices

9.2 (pg. 506) #1-11 odd
9.3 (pg. 515) #1, 3, 5, 9, 11, 13, 17, 19, 21, 23, 25, 27, 29, 31, 33, 39
9.4 (pg 523) #1, 9, 11, 13, 15, 17, 19
9.5 (pg. 534) #1, 3, 5, 7, 11, 13, 19, 21

Review Taylor Polynomials and Series

Ex 1) Find the Taylor polynomial of degree 5, centered at 0, for $f(x) = e^x$

Use $\sum_{n=0}^{5} \frac{f^{(n)}(c)}{n!}(x-c)^n$

$f^{(n)}(x)$	$f^{(n)}(c)$	$n!$	$(x-c)^n$

Thus, $e^x \approx$

By the Taylor polynomial, $e^{0.1} \approx$

By calculator $e^{0.1} \approx$

Ex 2) Find the Taylor Polynomial for sin(x) centered at $x = \dfrac{\pi}{2}$ to 3rd degree.

Common Series (there are more than this):

$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots$	$R = \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \dfrac{k(k-1)}{2!}x^2 + \dfrac{k(k-1)(k-2)}{3!}x^3 + \ldots$	$R = 1$

Before we apply this to differential equations, we need to be able to do the following:

If f(x) = $\sum_{i=0}^{\infty} a_n x^n$, then find $f', f'', and\ f'''$

Re-index so that the sums start back at 0.

Solve $(x-3)y' + 2y = 0$ using Power Series:

Solve using Power Series: $y' - 2xy = 0$

Initial Value Problems and Power Series (Nice Trick compared to previous pages).
Find a 3rd degree Taylor Polynomial for the IVP $y'' - 2xy' + x^2 y = 0$, $y(0) = 1$, $y'(0) = -1$

Find a 4th degree Taylor Polynomial for the given IVP $y'' - 2xy = 0$, $y(0) = 3$, $y'(0) = 4$.

Using Maple to find a power series solution of a differential equation.
As you have seen, solving a differential equation using power series can be tedious. Now when the purpose is to solve a problem and a suitable device is available, it is often useful to know how to use the device.

You can do your own searches, if and when you have time, here are a few examples to get you started.

To solve $\frac{dy}{dx} = y$ say, using power series you punch in the following commands:

```
> with(powseries):
> a:=powsolve(diff(y(x),x)=y(x)):
> tpsform(a, x,8);
```

On hitting enter you would get:

$$C0 + C0\,x + \frac{1}{2}C0\,x^2 + \frac{1}{6}C0\,x^3 + \frac{1}{24}C0\,x^4 + \frac{1}{120}C0\,x^5 + \frac{1}{720}C0\,x^6 + \frac{1}{5040}C0\,x^7 + \mathrm{O}(x^8)$$

Here C0 is a constant and the above line means that the solution is $y = C0(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + O(x^8))$ (Here $O(x^8)$ means that after $\frac{1}{5040}x^7$ there are terms that involve power 8 or higher of *x*.
If you do not give any number you would get the series truncated at the fifth power. Here is the example:

```
> a:=powsolve(diff(y(x),x)=y(x)):
> tpsform(a, x);
```

$$C0 + C0\,x + \frac{1}{2}C0\,x^2 + \frac{1}{6}C0\,x^3 + \frac{1}{24}C0\,x^4 + \frac{1}{120}C0\,x^5 + \mathrm{O}(x^6)$$

Before I let you play with it, I ask you to remember one thing. The software will not give you a general term of the power series. For example the power series solution given above does not give you the information that the solution of $\frac{dy}{dx} = y$ is your usual $y = C0e^x$.

Now here is a somewhat nontrivial example:

```
>with(powseries):
> a:=powsolve(diff(y(x),x$2)-y(x)=0):
> tpsform(a,x,8);
```

$$C0 + C1\,x + \frac{1}{2}C0\,x^2 + \frac{1}{6}C1\,x^3 + \frac{1}{24}C0\,x^4 + \frac{1}{120}C1\,x^5 + \frac{1}{720}C0\,x^6 + \frac{1}{5040}C1\,x^7 + \mathrm{O}(x^8)$$

Here there are two constants involved that means that there are two linearly independent solutions. Once you sift though you would find that the solutions are: $y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \ldots$ and $y_2 = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \ldots$

There is the last job of taking care of problems like
$y'' - xy' - y = 0, \quad x_0 = 0, \quad y(0) = 2, \quad y'(0) = 1$

For such problems the command is:

```
> a:=powsolve(diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(0)=2,D(y)(0)=1):
> tpsform(a,x,8);
```

$$2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \frac{1}{105}x^7 + O(x^8)$$

Practice from Textbook:
8.2 (pg. 434) #1, 3, 5, 29
8.3 (pg. 445) #25, 27, 29, 31

Mathematician, Physicist, Engineer walking through a field comes upon a farmer.
The farmer asks what is the best way to construct a fence that will contain his livestock (*ie*., most area for least perimeter). The physicist does some calculus and concludes that the best way to do this is a square fence. The engineer looks at him and laughs. "No, the best way is a circle". The physicist concedes and they start building the fence.
The mathematician just sits there for a while and eventually stands up, puts a small piece around him and says, "I declare myself to be outside".

Section 1.3 - Direction Fields (graphing ODE)

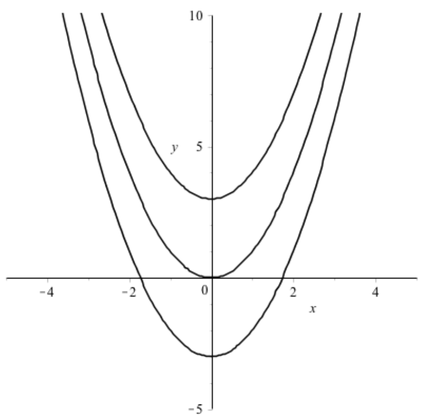
Goal: Visualize a differential equation and its solution.

Fact: In most real world cases, it is NOT possible to find a solution to an ODE.

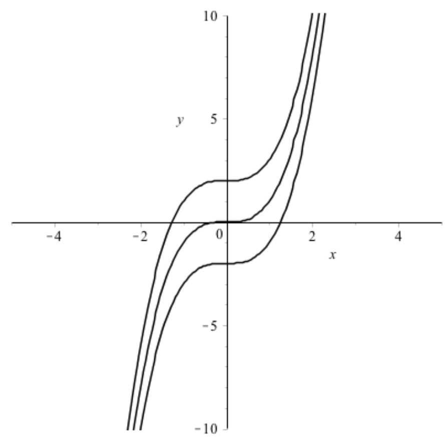
Try: Approximate the solution using direction fields (we will be graphing the differential equation and using it to approximate what the original equation is doing). I.E. We will be graphing a bunch of tangent lines (the derivative represents the slope of the tangent line)

Ex1) In your groups, draw tangent lines on the on each graph at x = -3, -2, -1, 0, 1, 2, 3:

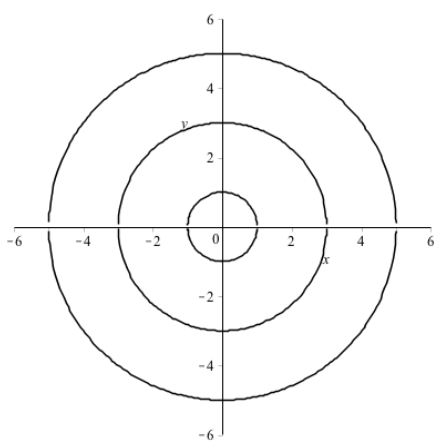
A.



B.



C. plot tangent lines at x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5



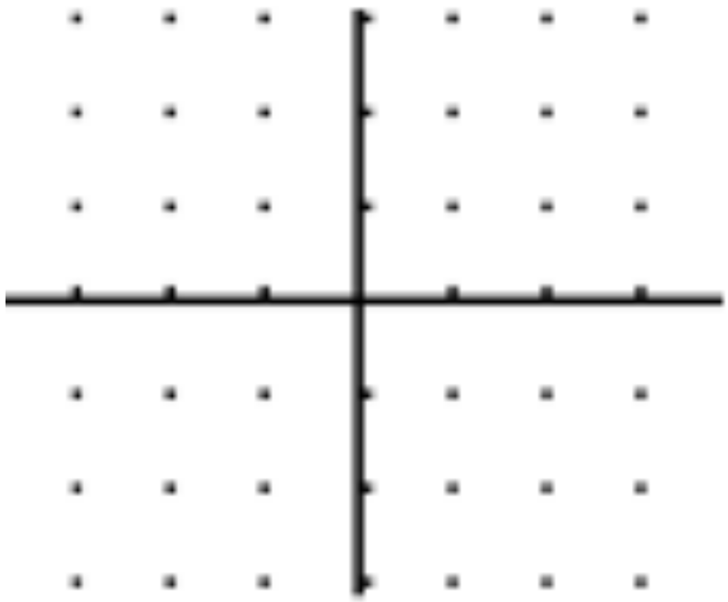
Ex2). At your tables, find the value of dy/dx = -x/y for the given table numbers. Plot the tangent line on the board.

Table 1: (-4, y) Table 2: (-3, y) Table 3: (-2, y)

Table 4: (-1, y) Table 5: (1, y) Table 6: (2, y)

Table 7: (3, y) Table 8: (4, y)

Mrs. B: (0,y)

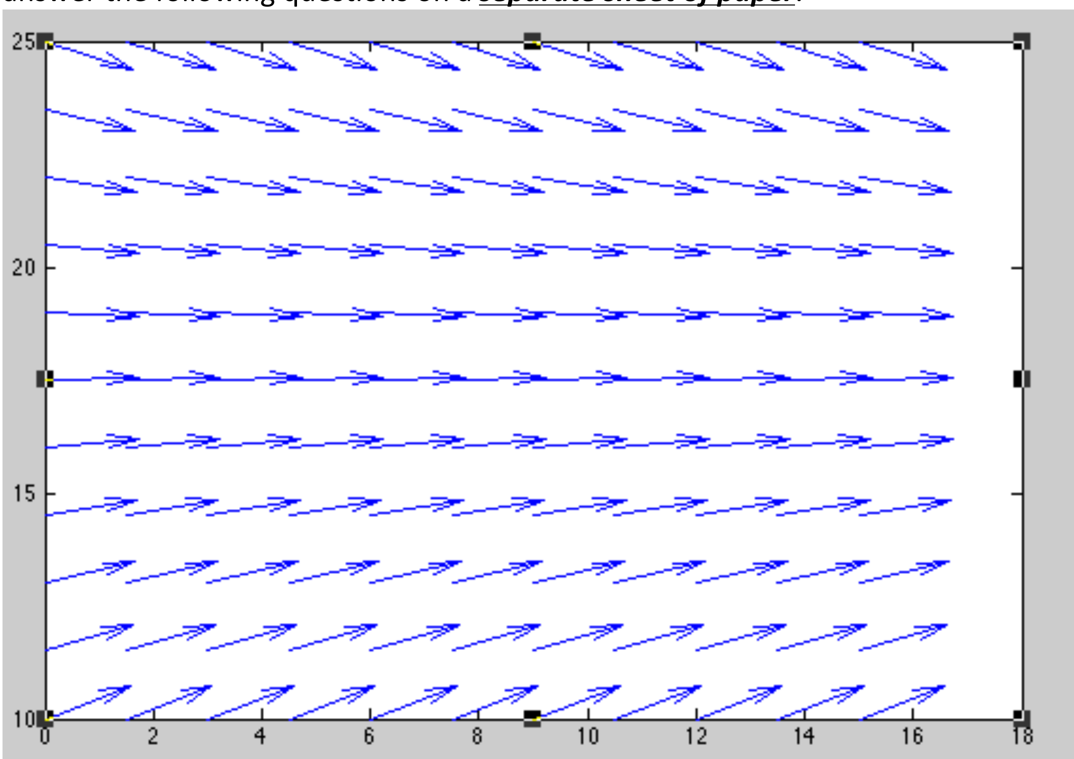


Based off of your graph above, what do you think is the solution to the ODE?

A slope field (direction field) shows the general "flow" of a differential equation's solution. They can be used to approximate solutions.

Ok – now let's suppose we give you a direction field (i.e. graph of a bunch of tangent lines that represent the family of functions that solve a differential equation). Answer the following based off of the given direction field.

Suppose you were in a science class, and you were given the following direction field where the independent variable is time and the dependent variable is the temperature (in Celsius) of some liquid that is held in a glass jar. Based off of the given direction field, answer the following questions on a ***separate sheet of paper***.



A. If the temperature of the liquid is 13 degrees Celsius, what would be the terminal temperature? (I.E. what would be the temperature if you left the liquid in the room for a very long time). EXPLAIN your reasoning for your answer.

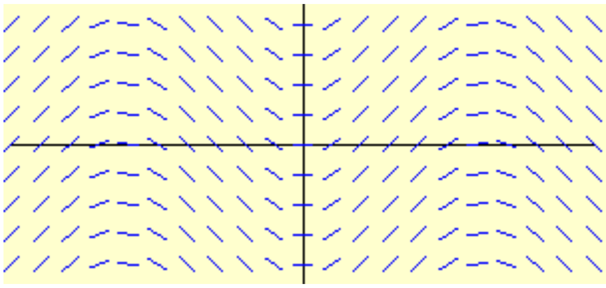
B. Sketch the solution to the ODE that has an initial temperature of 13 degrees Celsius on the graph above (clearly label your graph). EXPLAIN your reasoning for your answer.

C. If the temperature of the liquid is 25 degrees Celsius, what would be the terminal temperature? Draw the solution to the ODE that has an initial temp of 25 degrees Celsius on the graph above (clearly label your graph). EXPLAIN your reasoning for your answer.

When you are done, turn in this page along with your work from the separate sheet of paper. 5pts hwk

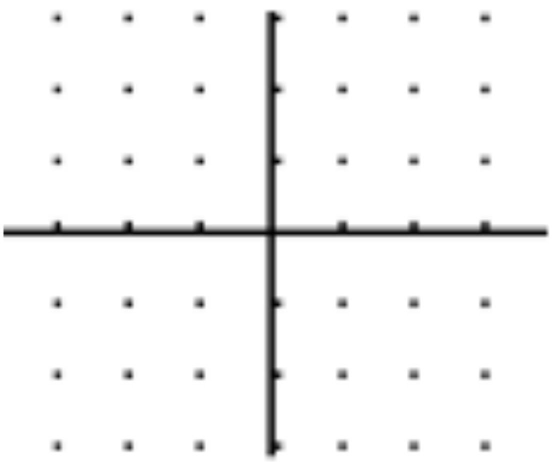
Ex 3) Let f(t,y)=dy/dt = sint then what is y?

Below is the graph of dy/dt. Does the slope field match your solution to y?



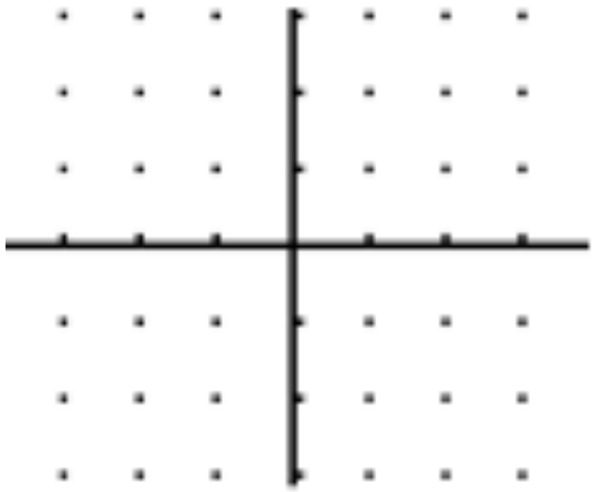
At your tables, draw a slope field (direction field) for the following differential equations. Answer the given questions that following your slope field.

4. $\dfrac{dy}{dt} = t + 1$



A. If y(1) = -1, draw the solution curve. Estimate y(0) based off of your solution curve.

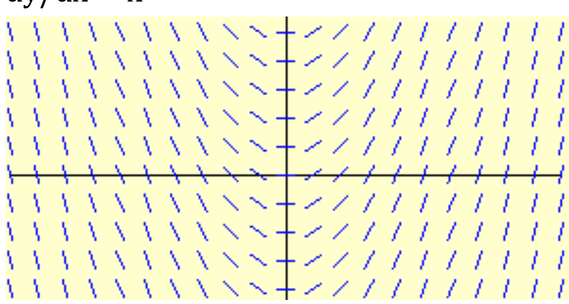
5. $\frac{dy}{dx} = -\frac{y}{x}$



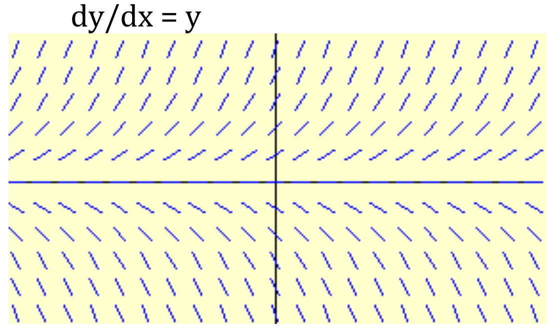
A. If y(0)=-2, draw your solution curve. Estimate y(1) based off of your solution curve.

More Examples:

dy/dx = x

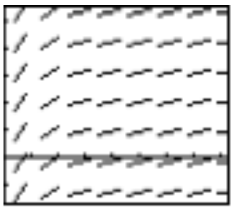


dy/dx = y



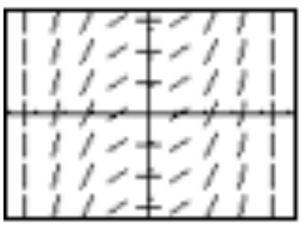
If y(-2) = 1, draw the solution curves for both ODE above.
Estimate y(0) based off of your curve.

6. The slope field from a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?



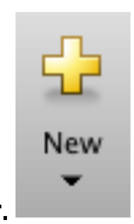
(A) $y = x^2$ (B) $y = e^x$ (C) $y = e^{-x}$ (D) $y = \cos x$ (E) $y = \ln x$

7. The slope field for a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?



(A) $y = \sin x$ (B) $y = \cos x$ (C) $y = x^2$ (D) $y = \frac{1}{6}x^3$ (E) $y = \ln x$

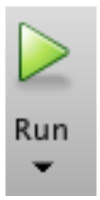
Now, let's use MATLAB to graph direction fields.

1st: Open MATLAB and click on "New" in top left hand corner. 

Then click on "script".
A new window should open up that says "Editor-untitled". Here is where you will create an "mfile". You will be able to save this file and use it on exams.

2nd: Type in the following into the script (Note: You can choose any notes to insert after the % symbol).

```
%This matlab file will draw direction fields for the provided DE
%Save file: dirfields.m
%To run and see other DE, then change the dY equation
t=-3:1:5; y=10:1:25;              %Edit viewing window here
[T,Y]=meshgrid(t,y);
dT=ones(size(T));
dY=-0.055*Y+1;          %Insert ODE here - be careful with .* vs * only
N=sqrt(dT.^2+dY.^2);
dT=dT./N;
dY=dY./N;
axis equal; axis ([-4 4 -4 4]); grid on;
quiver(T,Y,dT,dY)
```

3rd: Click on the "Run" arrow: 

4th: A window will pop up to ask you to save this. Choose a name for your program (I chose "dirfields") Keep the "where" location as MATLAB. Next time you open MATLAB, this should be there for you to use on the next exam. (you can also save on your jump drive).

5th: Once the program is saved, you should see the direction field. =)

Click the run button, and you should see a direction field for
dy/dt = `=-0.055*Y+1`

Note: You can write your own direction field script and save it. There are much better ones than the one on the previous page.

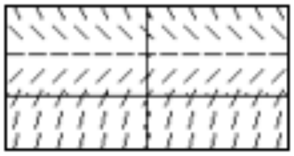
Be careful! If you have multiplication of variables (or division), you must use .* or ./

Also – this code requires the use of capital letters for your ODE to run (T and Y).

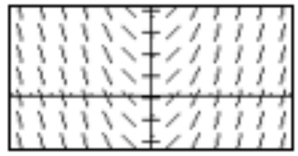
You can insert notes like this into your script and save it.

8. Match the slope fields with their differential equations.

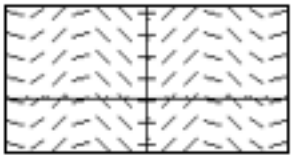
(A)



(B)



(C)



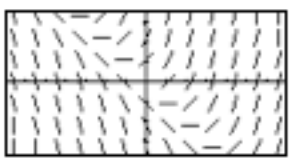
(D)

I. $\frac{dy}{dx} = \sin x$ II. $\frac{dy}{dx} = x - y$ III. $\frac{dy}{dx} = 2 - y$ IV. $\frac{dy}{dx} = x$

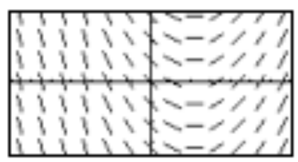
---

9. Match the slope fields with their differential equations.

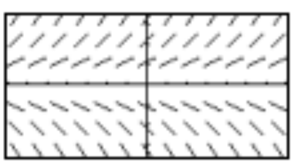
(A)



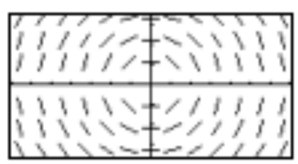
(B)



(C)



(D)



I. $\frac{dy}{dx} = .5x - 1$ II. $\frac{dy}{dx} = .5y$ III. $\frac{dy}{dx} = -\frac{x}{y}$ IV. $\frac{dy}{dx} = x + y$

10. (a) On the slope field for $\frac{dP}{dt} = 3P - 3P^2$, sketch three solution curves showing different types of behavior for the population P.

(b) Is there a stable value of the population? If so, what is it?

(c) Describe the meaning of the shape of the solution curves for the population: Where is P increasing? Decreasing? What happens in the long run? Are there any inflection points? Where? What do they mean for the population?

(d) Sketch a graph of $\frac{dP}{dt}$ against P. Where is $\frac{dP}{dt}$ positive? Negative? Zero? Maximum? How do your observations about $\frac{dP}{dt}$ explain the shapes of your solution curves?

For each situation in 11 and 12,

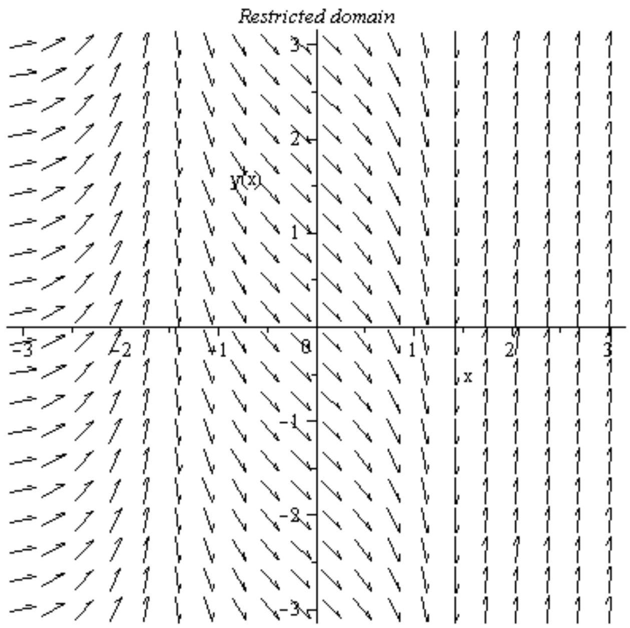
(a) Write a differential equation from the given information and answer the given question.

(b) Use MATLAB to sketch a slope field for the DE. Use the slope field to determine the long-term pattern. Include the solution curve on your direction field.

11. Little Red Riding Hood loses 20% of her muffins to the Big Bad Wolf on every mile of her walk to Grandma's house. The Muffin Man supplies her with 3 more every mile (he's sort of a stalker). How many muffins will Red Riding Hood eventually have for her Grandma if she starts with 6 muffins?

12. Goldilocks eats 75% of the Bears porridge every time she visits, but Mama Bear adds 5 oz of porridge to the pot after each visit. If Mama Bear started with a 16 oz porridge pot, how much porridge will Mama Bear eventually have?

13. Given the following direction field, answer the provided questions/statement.



A. Explain what the direction field represents.

B. Give at least 3 characteristics that the original equation may have.

C. If y(-3) = -2 is the initial value, what happens to y as x increases in value?

D. For what values of x does the original function increase most rapidly? Please explain.

Practice: 1.3 (pg. 21) #1-5, 7

1.4 Euler's Method

From PPT

$$y = y_0 + f(x_0, y_0)(x - x_0)$$

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

So only first order ordinary differential equations can be solved using Euler's method. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in $\frac{dy}{dx} = f(x, y),\ y(0) = y_0$ form.

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2 – You Try

Rewrite

$$e^{y}dy + x^{2}y^{2}dx = 2\sin(3x)dx,\ \ y(0) = 5$$

in $\dfrac{dy}{dx} = f(x, y),\ y(0) = y_0$ form.

From PPT (Note: h(0) = 0):

$$\frac{dh}{dt} = t$$

Step Size of h = 0.5

Step Size of h = 0.2

$$(x_1, y_1) = \begin{cases} x_1 = x_0 + \Delta x \\ y_1 = y_0 + f(x_0, y_0)\Delta x \end{cases}$$

$$(x_2, y_2) = \begin{cases} x_2 = x_1 + \Delta x \\ y_2 = y_1 + f(x_1, y_1)\Delta x \end{cases}$$

$$(x_3, y_3) = \begin{cases} x_3 = x_2 + \Delta x \\ y_3 = y_2 + f(x_2, y_2)\Delta x \end{cases}$$

$$(x_4, y_4) = \begin{cases} x_4 = x_3 + \Delta x \\ y_4 = y_3 + f(x_3, y_3)\Delta x \end{cases}$$

In general the coordinates of the point (x_{n+1}, y_{n+1}) can be computed from the coordinates of the point (x_n, y_n) as follows

$$(x_{n+1}, y_{n+1}) = \begin{cases} x_{n+1} = x_n + \Delta x \\ y_{n+1} = y_n + f(x_n, y_n)\Delta x \end{cases}$$

Practice:

1a. Given the differential equation dy/dx = x + 2 and y(0) = 3, Find an approximation for y(1) by using Euler's method with 5 equal steps.

1b. Solve the differential equation in part A and use your solution to find y(1).

1c. The error in using Euler's method is the difference between the approximate value and the exact value. What was the error in your answer? How could you produce a smaller error using Euler's Method?

MATLAB and Euler's Method

Open MATLAB and create a new script – save it as euler.m

The following will be in your script:

```
function [y,t]=euler(fun,y0,t0,T,h)
%[y,t] = euler(fun,y0,t0,T,h)-
%
%This function computes the solution to the IVP y'(t)=fun(y,t),
%y(t0)=y0, for a given function "fun(t,y)" using Euler's method
%
%The function is defined either via an m-file fun.m or using the
%"inline" command.  y0 is the initial value, T is the maximum time, h
%is the stepsize and t0=initial time.
%
%The output is a vector containing the approximate solution y_euler.
%
%To run this m-file - enter the following 1st!
%              f=inline('insert DE here','t','y')
%              [y,t]=euler(f,y0,t0,T,h)  (insert initial y and t value,
%                                  insert value to stop (T), and step size
(h)
%Once you enter those two commands, you will get euler's approx!  =)
y(1)=y0;
t(1)=t0;
for i=1:T/h
    y(i+1)=y(i) +h*feval(fun,t(i),y(i));
    t(i+1)=t(i)+h;
end;
t=t';
y=y';
%End of m-file euler.m
```

To demonstrate the use of the above m-file, consider solving the following IVP:

$$y'(x) = x\sqrt{y}\ ,\ y(1) = 4$$

for $x \in [1,2]$, using $h = 0.2$ and 0.1. We begin by defining the right hand side function $f(x,y) = x\sqrt{y}$, using MATLAB's `inline` command:

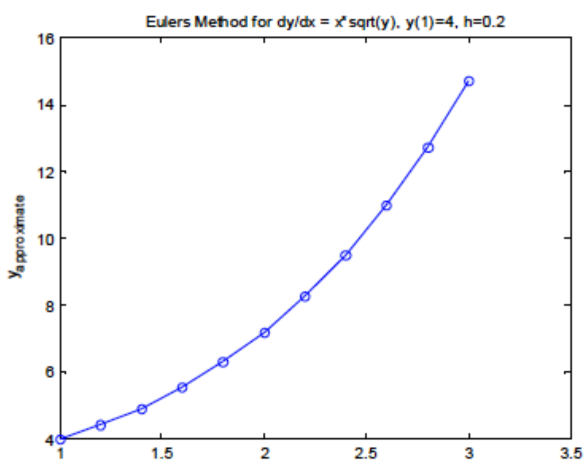
```
>> f = inline('x*sqrt(y)')

f =
     Inline function:
     f(x,y) = x*sqrt(y)
```

Then, we "call" the m-file as follows:

```
>> [y,t] = euler(f,4,1,2,0.2)

y =

    4.0000
    4.4000
    4.9034
    5.5235
    6.2755
    7.1774
    8.2490
    9.5127
   10.9931
   12.7173
   14.7143


t =

    1.0000
    1.2000
    1.4000
    1.6000
    1.8000
    2.0000
    2.2000
    2.4000
    2.6000
    2.8000
    3.0000
```

The above numbers are approximations to the true (x, y) points on the solution curve. We can plot the approximate solution as follows:

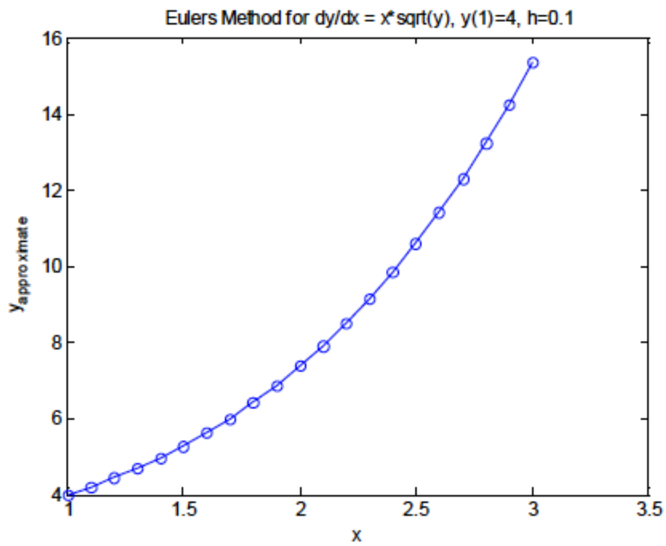
```
>> plot(x,y,'o-')
>> xlabel('x')
>> ylabel('y_{approximate}')
>> title('Eulers Method for dy/dx = x*sqrt(y), y(1)=4, h=0.2')
```



We can repeat the above steps using the other stepsize, $h = 0.1$:

```
>> [y,t] = euler(f,4,1,2,0.1)

y =
    4.0000
```

```
    4.2000
    4.4254
    4.6779
    4.9590
    5.2708
    5.6152
    5.9943
    6.4105
    6.8663
    7.3642
    7.9069
    8.4974
    9.1387
    9.8340
   10.5866
   11.4000
   12.2779
   13.2240
   14.2422
   15.3366


  =
    1.0000
    1.1000
    1.2000
    1.3000
    1.4000
    1.5000
    1.6000
    1.7000
    1.8000
    1.9000
    2.0000
    2.1000
    2.2000
    2.3000
    2.4000
    2.5000
    2.6000
    2.7000
    2.8000
    2.9000

    3.0000


>> plot(x,y,'o-')
>> xlabel('x')
>> ylabel('y_{approximate}')
>> title('Eulers Method for dy/dx = x*sqrt(y), y(1)=4, h=0.1')
```



Remark:

When the right hand side f depends only on one of the two variables, we need to tell MATLAB that in reality, f is a function of two variables. For example, to solve the IVP

$$\begin{cases} y'(t) = y(2-3) \quad , \quad 0 \le t \le 1 \\ y(0) = 3 \end{cases}$$

with, say $h = 0.25$, we define $f(t, y) = y(2-y)$ as follows:

```
>> f = inline('y*(2-y)', 't' , 'y' )
f =
     Inline function:
     f(t,y) = y*(2-y)
```

and call `euler.m` as before:

```
>> [y,t] = euler(f,3,0,1,0.25)

y =

    3.0000
    2.2500
    2.1094
    2.0517
    2.0252


t =

         0
    0.2500
    0.5000
    0.7500
    1.0000
```

Try the previous problems with MATLAB to verify that MATLAB and your work match.

Practice

(a) Write a differential equation from the given information.
(b) Sketch the slope field for the differential equation. The slope field should indicate clearly the long-term pattern. You will need to add axes to each slope field where you see fit for each problem.
(c) Include the solution curve, and answer the question based off of your solution curve.
(d) Use Euler's method to approximate the answer with a step size of 0.5 (since asked for long-term behavior – use MATLAB and euler.m to approximate solutions for large independent variable values)

1. Winnie the Pooh eats 75% of Rabbit's honey every time he visits, but Rabbit adds 5 ounces of honey to the pot after each visit. If Rabbit started with a 16 ounce honey pot, how much honey will Rabbit eventually have?

2. Winnie the Pooh is eating honey at a rate of 4 ounces/minute. If he starts with a 16 ounce pot of honey, when will he have no more left?

3. The 3 little pigs are building their houses. While the brick house building pig and the stick house building house pig are working at a steady rate, the straw house building pig (knowing the wolf will arrive first at his house) is frantically increasing his rate of building by 2ft/minute. The straw house building pig has to make his house 16 ft tall (right now, it is 0 ft tall) and the wolf is scheduled to arrive in 5 minutes. Will the pig have his house built for the wolf to blow down by the time he arrives?

Textbook practice: (pg. 28) 1.4 #1-7odd, 15

An engineer, mathematician, and physicist are each asked to determine the volume of a red metal ball. The mathematician measures the diameter, divides it by two to obtain the radius, and then performs a double integration.
The physicist weighs the ball and then weighs it again when immersed in water. Knowing the density of water and the difference in the two weights, she calculates the displaced volume of water, which is the volume of the ball.
The traditional engineer turns to her reference text The Physical Properties of Balls and in the chapter entitled "Metal", finds the table labeled "Red". Searching for a row that the contains the appropriate model number (which is stamped on the ball), she reads across to the column "volume", ignoring those dealing with "coefficient of thermal expansion" and "software rev. level".
The modern, 'Net savvy engineer' Google's for "The Volume of the Red Ball" and finds this page (http://www.farmdale.com/emp-jokes.shtml).

3.7 Runga Kutta

A PPT will be done in class covering this much more powerful way of approximating solutions to ODE.

MATLAB and Runga Kutta

Note: Once you have created the m-file for Runga Kutta – NEVER TOUCH THE M-FILE AGAIN! You type everything into command window.

Copy and paste the following into an m-file and save it as rk4

```
function [yi, ti] = rk4 ( RHS, t0, y0, tf, N )

%RK4      approximate the solution of the initial value problem
%
%                y'(t) = RHS( t, y ),    y(t0) = y0
%    NOTE:  RHS is solving "right hand side" of the equation....solve for
%    y'
%        using the classical fourth-order Runge-Kutta method - this
%        routine will work for a system of first-order equations as
%        well as for a single equation
%
%    calling sequences:
%          RHS=inline('insert equation here','t','y')
%          [yi, ti] = rk4 ( RHS, t0, y0, tf, N )
%    inputs:
%          RHS     string containing name of m-file defining the
%                right-hand side of the differential equation;  the
%                m-file must take two inputs - first, the value of
%                the independent variable; second, the value of the
%                dependent variable
%          t0     initial value of the independent variable
%          y0     initial value of the dependent variable(s)
%                if solving a system of equations, this should be a
%                row vector containing all initial values
%          tf     final value of the independent variable
%          N      number of uniformly sized time steps to be taken to
%                advance the solution from t = t0 to t = tf
%
```

```
%    output:
%          yi     vector / matrix containing values of the approximate
%                 solution to the differential equation
%          ti     vector containing the values of the independent
%                 variable at which an approximate solution has been
%                 obtained
%

neqn = length ( y0 );
ti = linspace ( t0, tf, N+1 );
yi = [ zeros( neqn, N+1 ) ];
yi(1:neqn, 1) = y0';

h = ( tf - t0 ) / N;

for i = 1:N
   k1 = h * feval ( RHS, t0, y0 );
          k2 = h * feval ( RHS, t0 + h/2, y0 + k1/2 );
          k3 = h * feval ( RHS, t0 + h/2, y0 + k2/2 );
          k4 = h * feval ( RHS, t0 + h, y0 + k3 );
   y0 = y0 + ( k1 + 2*k2 + 2*k3 + k4 ) / 6;
          t0 = t0 + h;

   yi(1:neqn,i+1) = y0';
end;
```

Be able to do ALL the problems from Euler's approximation using Runga-Kutta in Matlab. Compare your results.

Section 2.2 – Separable Differential Equations

Separating Variables

A first-order differential equation in x and y is separable if it can be written so that all the terms involving y are on one side and all the terms involving x are on the other.

A differential equation has variables separable if it is of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \text{or} \quad g(y)\,dy - f(x)\,dx = 0.$$

The general solution is

$$\int g(y)\,dy - \int f(x)\,dx = C \qquad \text{(C an arbitrary constant)}$$

An Example

Problem Statement: Solve $y' = \dfrac{x^2}{y\left(1+x^3\right)}$.

Notice that $y \neq 0$ and $x \neq -1$ and $\dfrac{dy}{dx} = \dfrac{x^2}{y\left(1+x^3\right)} = \left(\dfrac{1}{y}\right)\left(\dfrac{x^2}{1+x^3}\right)$.

Step 1: Separate the variables.

$$ydy = \frac{x^2}{1+x^3}dx$$

Step 2: Integrate both sides of the equation from Step 1.

$$\int ydy = \int \frac{x^2}{1+x^3}dx = \frac{1}{3}\int \frac{du}{u} \qquad [\text{Let } u = 1+x^3, du = 3x^2dx]$$

$$\frac{1}{2}y^2 = \frac{1}{3}\ln|u| = \frac{1}{3}\ln\left|1+x^3\right| + C$$

So the one-parameter family is

$$\boxed{\frac{1}{2}y^2 = \frac{1}{3}\ln\left|1+x^3\right| + C},$$

Or multiplying through by 2 and setting *K=2C*,

$$\boxed{y^2 = \frac{2}{3}\ln\left|1+x^3\right| + K}.$$

An IVP (initial value problem) Example

Problem Statement: Solve the IVP $y' = 2(1+x)(1+y^2), y(0) = 0$.

Step 1: Separate the variables.

$$\frac{dy}{1+y^2} = (2+2x)\,dx$$

Step 2: Integrate both sides of the equation from Step 1.

$$\int \frac{dy}{1+y^2} = \int (2+2x)\,dx$$

$$\tan^{-1} y = 2x + x^2 + C$$

Step 3: **(only if an IVP is given)** Evaluate C using the initial condition.
To solve the IVP, apply the initial condition $y(0) = 0$:

$$\tan^{-1} 0 = 2(0) + 0^2 + C = C$$

$$C = 0$$

So the solution is $\boxed{\tan^{-1} y = 2x + x^2}$, or, taking the tangent of both sides,

$$\boxed{y = \tan\left(2x + x^2\right)}.$$

From PPT: Solve the following.

1. $dy - 5\sqrt{y-3}\sin(x)dx = 0,\ y(\pi) = 4$

2. $\dfrac{dr}{dt} = 5t^3 e^{-4r}, \quad r(1) = 2$

From PPT: Mixing problem

A brine solution of salt flows at a constant rate of 8L/min into a large tank that initially held 100L of brine solution in which was dissolved 0.5kg of salt. The solution inside the tank is kept well stirred and flows out at the same rate. If the concentration of salt in the brine entering the tank is 0.05 kg/L, determine the mass of salt in the tank after t min.

Practice (From Kuta Software)

Find the general solution of each differential equation.

1) $\frac{dy}{dx} = e^{x-y}$

2) $\frac{dy}{dx} = \frac{1}{\sec^2 y}$

3) $\frac{dy}{dx} = xe^{y}$

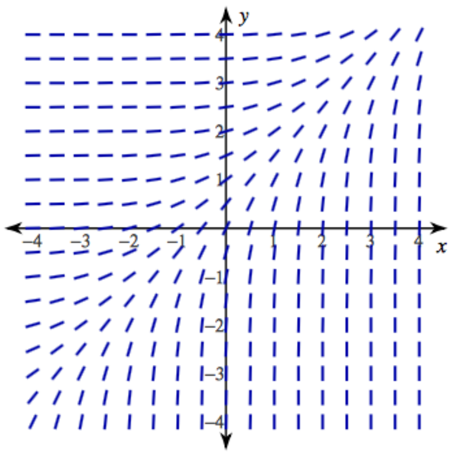
4) $\frac{dy}{dx} = \frac{2x}{e^{2y}}$

5) $\frac{dy}{dx} = 2y - 1$

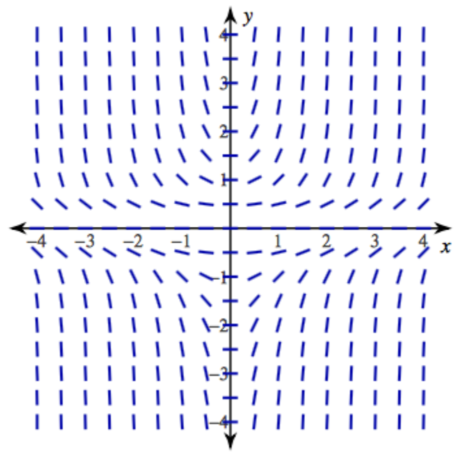
6) $\frac{dy}{dx} = 2yx + yx^2$

For each problem, find the particular solution of the differential equation that satisfies the initial condition. You may use a graphing calculator to sketch the solution on the provided graph.

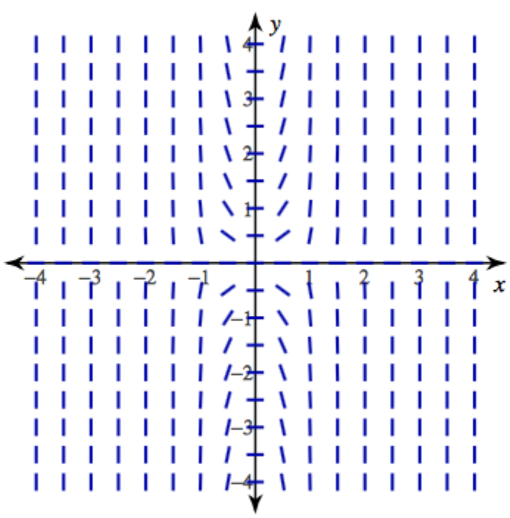
7) $\dfrac{dy}{dx} = 2e^{x-y},\ y(1) = \ln(2e+1)$



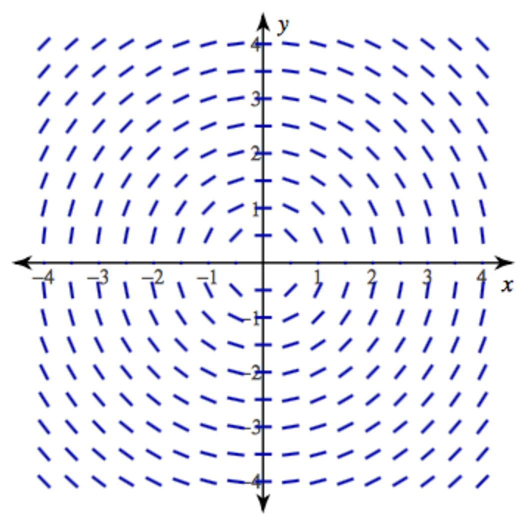
8) $\dfrac{dy}{dx} = xy^2,\ y(2) = -\dfrac{2}{5}$



9) $\dfrac{dy}{dx} = 12x^3y,\ y(0) = 2$



10) $\dfrac{dy}{dx} = -\dfrac{x}{y},\ y(1) = -\sqrt{2}$



Solutions to previous page

Find the general solution of each differential equation.

1) $\dfrac{dy}{dx} = e^{x-y}$

$e^y = e^x + C$

$y = \ln\left(e^x + C\right)$

2) $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

$\tan y = x + C$

$y = \tan^{-1}\left(x + C\right)$

3) $\dfrac{dy}{dx} = xe^y$

$-e^{-y} = \dfrac{x^2}{2} + C_1$

$y = -\ln\left(-\dfrac{x^2}{2} + C\right)$

4) $\dfrac{dy}{dx} = \dfrac{2x}{e^{2y}}$

$\dfrac{e^{2y}}{2} = x^2 + C_1$

$y = \dfrac{\ln\left(2x^2 + C\right)}{2}$

5) $\dfrac{dy}{dx} = 2y - 1$

$\dfrac{\ln\left|2y-1\right|}{2} = x + C_1$

$y = \dfrac{Ce^{2x} + 1}{2}$

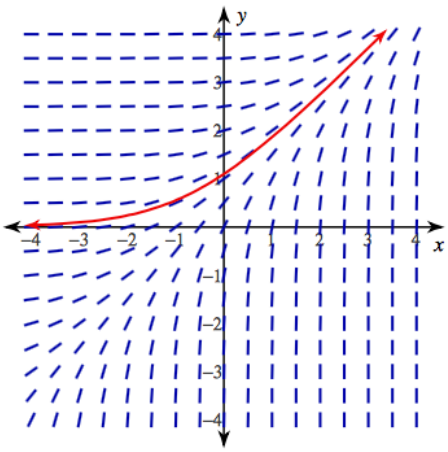
6) $\dfrac{dy}{dx} = 2yx + yx^2$

$\ln\left|y\right| = x^2 + \dfrac{x^3}{3} + C_1$

$y = Ce^{x^2 + \frac{x^3}{3}}$

For each problem, find the particular solution of the differential equation that satisfies the initial condition. You may use a graphing calculator to sketch the solution on the provided graph.

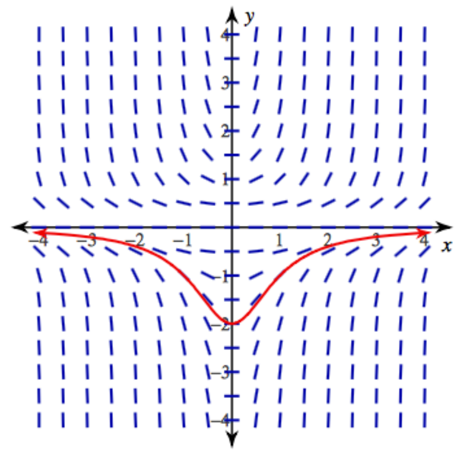
7) $\dfrac{dy}{dx} = 2e^{x-y},\ y(1) = \ln\left(2e+1\right)$



$e^y = 2e^x + 1$

$y = \ln\left(2e^x + 1\right)$

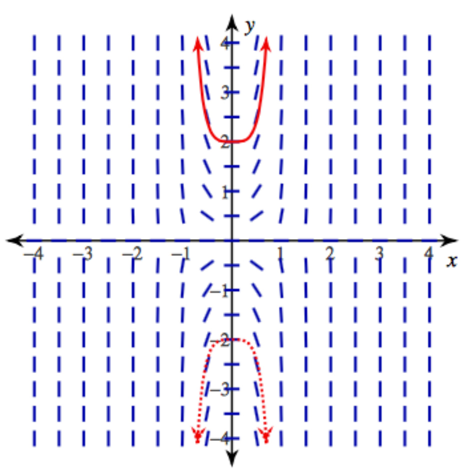
8) $\dfrac{dy}{dx} = xy^2,\ y(2) = -\dfrac{2}{5}$



$-\dfrac{1}{y} = \dfrac{x^2}{2} + \dfrac{1}{2}$

$y = -\dfrac{2}{x^2+1}$

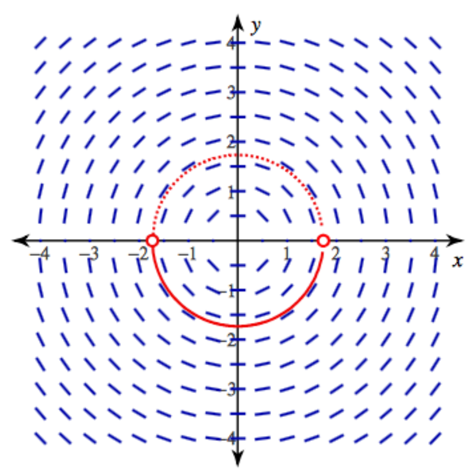
9) $\dfrac{dy}{dx} = 12x^3y,\ y(0) = 2$



$\ln\left|y\right| = 3x^4 + \ln 2$

$y = 2e^{3x^4}$

10) $\dfrac{dy}{dx} = -\dfrac{x}{y},\ y(1) = -\sqrt{2}$



$\dfrac{y^2}{2} = -\dfrac{x^2}{2} + \dfrac{3}{2}$

$y = -\sqrt{-x^2+3},\ -\sqrt{3} < x < \sqrt{3}$

Practice from textbook: (pg. 43) Section 2.2 #1-25 odd, 33, 34, 35, 37

Section 2.3 Linear Equations

From PPT:
Find derivative of the following:
y=sin(x)cos(x)

From PPT:
Solve ODE by using linear method. Then solve ODE by using separable equations (if possible).

$$y' + 2xy = x$$

From PPT:
Solve using linear method. Then solve by separable (if possible).

$$y' + \frac{1}{x}y = \sin x$$

$$y(\pi) = 1$$

Practice from textbook: (pg. 51) Section 2.3 #1-21odd, 35

An engineer, a mathematician, and a physicist are each sentenced to die by the guillotine. As the physicist is led to the guillotine, she decides that she'd like to observe the blade as it falls, perhaps to verify *v=at*, and she requests to be strapped in face up. The executioner agrees (why not? it all pays the same...), and straps her in. As the blade falls, it sticks about two thirds of the way down. Seeing this, the crowd cheers - the physicist must be innocent! So the executioner unstraps her and sets her free.
The mathematician is next. Being well versed in matters statistical (perhaps she is an actuary), she quickly asks to be placed face up as well - after all, the odds of it happening again are pretty good, especially if the initial conditions are similar. So the executioner obliges, once again, the blade sticks about two thirds of the way down. The crowd cheers, and the mathematician is also set free.
Finally, the engineer, not willing to do anything in public that is different from her peers, she, too, requests to be placed face up. As the executioner is strapping her in, she's looking up at the blade and studying the track in which it slides. As she does so, she notices something. "Do you see that?" she asks. "About one third the way up? If you fixed that there..."

Section 2.4 Exact Equations

See PPT

Practice from Textbook (pg. 61) 2.4 #1-25odd

Thermometer Accuracy - Lab

Need 4 groups

- Time Keeper
- Recorder of data (time and temperature of thermometer)
- Volunteer from each group to do: (a) chew gum vigorously (2 minutes) (b) drink cold water (2 minutes) (c) do jumping jacks (2 minutes)
- Person to shake thermometer until reading is below 94 degrees – this will be your initial condition, T_0 (shake for about 1-2 minutes) – do this while volunteers are doing their activities in bullet 3
- Person to input data into Logger Pro and run stats analysis

Differential Equation to use:

$$\frac{dT}{dt} = k(T - T_E)$$

T – Temperature of person's body at any given time
t – Time (you choose appropriate units)
T_E - Temperature constant (typically room temperature - but for this experiment, it will be the stabilized body temp - I.E. The end temperature of the body after 4 minutes)
k – Proportional constant

Experiment:

- Bring up Logger Pro (under Vernier) on laptops
- For 2 minutes, 1 volunteer from each group performs activity AND another group member shakes thermometer
- After 2 minutes, time keeper calls "Time" – read thermometer and record temp (this will be initial temperature at time 0)
- Record temperature and as soon as possible, insert thermometer in mouth of volunteer who performed activity (do not talk with thermometer in mouth)
- After 30 seconds – record temperature of thermometer (while it is in the mouth) – repeat this process each 30 seconds for at least 4 minutes in total.
- Record your data into Logger Pro – analyze data – highlight your data points and find the curve that best fits your data (try linear, quadratic, e^x and ln(x)) – which of these gave a best fit (look at correlation coefficient)

Lab Report (the following needs to be written as a lab report):

- Provide your data points and the equation of the best fit lines for linear, quadratic, e^x and ln(x). Include the correlation coefficient for each of these. Based off of your correlation coefficient, which one is the best fit line?
- Looking at your data, how long until the temperature stabilized?
- Use the differential equation and solve it. Keep all variables in problem to give a general equation.
- Determine your constant, k, using initial condition and another data point from your data.
- What is your solution to the ODE? How does it compare to your line that best fits the data? Are they the same or different? If different, why do you think this happened?
- According to your differential equation, how long until the temperature stabilized? I.E. How long does the thermometer need to stay under the tongue to get an accurate reading of your body temp? Did your data support this (see 2nd bullet)?

Send 1 write up (per group) to Mrs. B (email your write up). Make sure to include the names of everyone that was in your group.

3.2 Compartmental Analysis
Applications to Linear/Separable ODE

See PPT

Practice: 3.2 pg. 104 #1, 3, 5, 7, 19

Section 4.2, 4.3 and 4.4
See PPT

Section 4.1 – Mass Spring Systems

The PPT will cover a major application of section 4.2-4.4. We will be doing a lab with this concept.

We will be verifying that the following:
1. Path of a mass-spring system follows a sine wave
2. Investigate the different quantities that impact the movement of the spring
3. Show that a mass spring system can be modeled by a 2^{nd} order differential equation (see below)

$$my'' + by' + ky = -F_{ext}$$

4. Show a real life use for the imaginary number, i

We need m (mass), b (friction coefficient) and k (spring constant – stiffness of spring).

Need 4 people per group
In each group, there must be a person who is familiar with the following (students who have taken physics here at EMCC should be familiar with these)

- Logger Pro – Performing video analysis and using Force Probes
- Maple
- Excel
- Person with IPhone to take video of spring (if this is not possible, we can borrow cameras from the physics class)

General Equipment (* items will vary between groups)
- IPhone (or camera)
- Logger Pro
- Maple
- Excel
- *Spring
- *Mass
- Hook to hang springs
- *Plate
- Force probes

Items for each group:

Group 1 • IPhone (or camera) • Logger Pro • Maple • Excel • Short Spring • 20g Mass • Hook to hang springs • Force Probe	Group 2 • IPhone (or camera) • Logger Pro • Maple • Excel • Short Spring • 20g Mass • Hook to hang springs • Plate • Force Probe
Group 3 • IPhone (or camera) • Logger Pro • Maple • Excel • Long Spring • 20g Mass • Hook to hang springs • Force Probe	Group 4 • IPhone (or camera) • Logger Pro • Maple • Excel • Long Spring • 20g Mass • Hook to hang springs • Plate • Force Probe
Group 5 • IPhone (or camera) • Logger Pro • Maple • Excel • Short Spring • 50g Mass • Hook to hang springs • Force Probe	Group 6 • IPhone (or camera) • Logger Pro • Maple • Excel • Short Spring • 50g Mass • Hook to hang springs • Plate • Force Probe
Group 7 • IPhone (or camera) • Logger Pro • Maple • Excel • Long Spring • 50g Mass • Hook to hang springs • Force Probe	Group 8 • IPhone (or camera) • Logger Pro • Maple • Excel • Long Spring • 50g Mass • Hook to hang springs • Force Probe

Experiment:

Step 1: Stiffness of the spring

- Each group will need to determine the stiffness of the spring (find the k value for the spring constant)
- Need Force probes, Logger Pro

For elastic springs (which we are using), the relationship between the applied force and the distance the spring moves is called Hooke's Law.

$F = -kx$

******Notice that k is the slope of the linear equation.

- Open Logger Pro

- Connect the Force Probe to the laptop (Logger Pro should recognize the probe)

- Attach spring to hook, zero out Logger Pro
- Hit the "collect" button and begin to stretch the spring at a CONSTANT rate
- You should see a linear graph
- Determine the spring constant (see the starred notice above)

Run experiment 2 more times to verify that your spring constant is accurate.

You have now found the "k" in the differential equation.

$$my'' + by' + ky = -F_{ext}$$

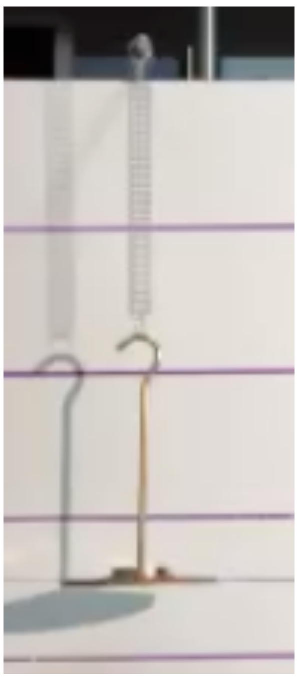
Step 2: Video Analysis – There MUST BE at least 1 student in each group that has done this in physics.

You will need the following:

- Meter Stick
- White board marker
- White Board
- Spring (short or long, depending on your group)
- Mass (20g or 50g, depending on your group)
- Plate (for those groups who are using a plate – tape plate to bottom of mass)
- Hook to hang spring and mass

You will perform video analysis of the spring.

- Hold top of spring, put the hook with mass at bottom of spring



- Have meter stick next to your spring (all should be in front of white board)
- Hold spring so that it is at equilibrium (see picture above)
- Take video of spring at rest, and then stretch spring – let go. Video the oscillations of the spring.
- Upload Video to Logger Pro and do video analysis
- Determine the curve that best fits the motion of the spring.

Does your best-fit line result in a sine wave?

Step 3: Determining the minimum value for the force of friction coefficient, b
Maple will be needed, along with initial conditions.

- Everyone will have an initial condition of $y'(0) = 0$, since the mass spring started at rest (zero velocity).
- Determine y(0) based off of your best fit line (note: your answer should be close to zero, but it won't be exactly zero).
- Open Maple – click on "New Worksheet".
- Note: The external force acting on the spring was gravity – so we will use 10 m/s/s as our gravitational constant
- Insert the following commands to Maple:

```
>  ode:= y^(2) (x) + b/m *y^(1)(x) + k/m *y(x) =-10
>ics:=y(0) = (insert your group's initial condition value here), D(y)(0)=0
> spring:=dsolve({ode,ics},y(x))
```

You will have a very ugly equation.

Question: What are the above commands doing in Maple and AND what do they represent? I.E. Interpret the Maple codes above. (There are two things occurring in each line of command).

From here, you will put in your initial conditions of your mass and spring constant.
20 g mass – use 0.02 (why?)
50 g mass – use 0.05 (why?)
k – each group will have a different value for k based off of your Step 1 data.

```
>subs(m = (insert mass), k = (insert your spring constant), spring);
```

Your very ugly equation will reduce down, but still be ugly.

Questions: What does the Maple code do? What value are we missing?

Since Logger Pro showed us that our best-fit line is a sine curve, what is the ONLY way for our 2nd order differential equation solution to have a sine function as part of the answer? I.E. What does the sign under the radical have to be in order to get a sine function as part of our solution?

Looking at your equation in Maple and the expression under the radical, find the smallest value of b so that you get a sine function?

Using Maple, plug in this b value to show that your solution is of the correct form. What does Maple give back as an answer?

Step 4:

Goals

- Use Excel to help “fine tune” the b value
- Show the 2nd order differential equation does model a mass spring system
- Show real life use of complex number, i

Equipment

- Excel
- Time data from logger pro
- Paper and Pencil
- Maple

Using your m, k and smallest value found in step 3 for b, plug them into your ODE below. Recall from section 4.4 that we must 1st solve the ODE by finding homogenous solutions 1st. Do this. Compare what you have to your Maple solution. What do you notice?

$$my'' + by' + ky = -F_{ext}$$

So, what would be our homogeneous solution?

Go back to your data in Logger Pro, and highlight the time data for the interval in which you found your equation of the best-fit line. Copy this time data and insert it into Excel, column 1.

In column 2, input your solution to the differential equation with the time value from column 1 (more will be explained in class).

Graph the two columns. Input different values for b until you get a sinusoidal graph.

Done!

Lab Report, to be submitted for a grade:

- Send an email to Mrs. B with the following attachments:
  - Maple
  - Excel
  - Logger Pro
- Word document answering the questions asked in each step of the entire lab experiment. If you do not have equation editor, you can submit a typed Word document with handwritten work for the math equations.
- In your own words, summarize the lab and purpose of it. Summarize the findings and what was determined through this experiment.

Send 1 write up (per group) to Mrs. B. Make sure to include the names of everyone that was in your group.

Chapter 7 – Laplace Transform

See PPT

a. <u>Calculate the Laplace Transform using Matlab</u>

Calculating the Laplace **F**(s) transform of a function f(t) is quite simple in Matlab. First you need to specify that the variable t and s are symbolic ones. This is done with the command

```
>> syms t s
```

Next you define the function f(t). The actual command to calculate the transform is

```
>> F=laplace(f,t,s)
```

To make the expression more readable one can use the commands, `simplify` and `pretty`.

here is an example for the function f(t),

Do the following using MATLAB.

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

```
>> syms t s
>> f=-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t);
>> F=laplace(f,t,s)

F =

-5/4/s+7/2/(s+2)^2+5/4/(s+2)

>> simplify(F)

ans =

(s-5)/s/(s+2)^2

>> pretty(ans)
                           s - 5
                         ----------
                                   2
                          s (s + 2)
```

which corresponds to F(s),

$$F(s) = \frac{(s-5)}{s(s+2)^2}$$

Alternatively, one can write the function f(t) directly as part of the laplace command:

```
>>F2=laplace(-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t))
```

b. <u>Inverse Laplace Transform</u>

The command one uses now is `ilaplace`. One also needs to define the symbols t and s. Lets calculate the inverse of the previous function F(s),

$$F(s) = \frac{(s-5)}{s(s+2)^2}$$

```
>> syms t s
>> F=(s-5)/(s*(s+2)^2);
>> ilaplace(F)
ans =
-5/4+(7/2*t+5/4)*exp(-2*t)
>> simplify(ans)
ans =
-5/4+7/2*t*exp(-2*t)+5/4*exp(-2*t)
>> pretty(ans)
        - 5/4 + 7/2 t exp(-2 t) + 5/4 exp(-2 t)
```

Which corresponds to f(t)

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

Alternatively one can write

```
>> ilaplace((s-5)/(s*(s+2)^2))
```

MATLAB Quick Notes for Laplace

Laplace Transform:

```
>> syms t s
>> f = (insert your function here)
>>F = laplace(f,t,s)
>>simplify(F)
>>pretty(ans)
```

Inverse Laplace Transform:

```
>>ilaplace(F)
```

Heaviside Functions (step functions)

```
>> heaviside(t-#)
```